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Polarization Planning for Wireless Networks

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Abstract Polarization diversity enables frequency reuse in a telecommunication network. The most widely considered solution is to use two orthogonal polarizations on the same link, which enables to double the available bandwidth. In this paper, we study the possibility to connect the nodes of a ring topology network with one single channel for all the links, with the condition that the polarization of any link is orthogonal to the polarization of the two adjacent links. The solution proposed in this paper can improve spectrum efficiency by up to 50% in comparison with the widespread polarization multiplexing solution. Furthermore, it has implications on network topology and channel allocation.

Keywords Polarization · Frequency reuse · Ring networks

1 Introduction

The wireless industry is presently facing a tremendous growth of demand for higher data rates, driven by the development of mobile data services. In order

Part of the material in this paper was presented in [4].

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to meet this demand, a number of solutions are widely considered: network densification (including deployment of small cells [15], [1], [7]), use of wider spectrum [15], [1], increase of spectral efficiency [15], [7].

Polarization diversity has long been regarded as another mean to meet the growing demand for wireless spectrum [11], [14]. This solution is suitable for line-of-sight (LOS) links only, because of the cross-polarization interferences resulting from reflections. Since conventional networks used only vertical polarization (VP), polarization diversity was achieved by adding a horizontal polarization (HP). Because of this historical context, polarization diversity was, in a large majority of cases, restricted to horizontal/vertical.

Since then, other options have been considered. The respective performances of HP/VP versus ±45° slanted polarization have been compared, in order to determine the optimal pair of orthogonal polarizations [20], [12], [3]. These works came along with the increased interest polarization diversity has gained these recent years in frequency planning.

Polarization multiplexing in non-LOS channels has been studied in [10]. However, the proposed solution is relevant only for one-way transmission signals, since there is no reversibility principle for polarization. For this reason, the LOS assumption is necessary for polarization multiplexing in full-duplex links. This assumption could be considered as very restrictive, especially in the context of the multiple-input multiple-output (MIMO) technology: the worst case scenario occurs for compact arrays under LOS propagation conditions, where the rank of the channel matrix equals to 1 [16], [13]. However, recent works have highlighted that millimeter wave is the most relevant solution for massive MIMO, due to the large available bandwidth [19] and the possibility to pack many antenna elements within a limited volume [22]. Due to the highly directional and quasi-optical nature of millimeter wave propagation, LOS propagation condition should be dominant in MIMO systems with millimeter wave communications [18], [2], [17].

Polarization multiplexing is presently seen as a method for doubling spectral efficiency on a single channel [6]. To the best of our knowledge, network coverage with one single channel by just optimizing the polarization directions has not been studied yet. In this paper, we show how an appropriate use of polarization diversity in ring topology networks improves spectrum efficiency.

While the current approach focuses on optimizing a point-to-point link, we propose a paradigm shift for polarization diversity: instead of restricting the choice to two pre-defined polarizations such as horizontal/vertical or ±45°, our approach, which will be called hereafter "inclined polarization", is based on polarizations fulfilling the condition that the polarization of any link is orthogonal to the polarization of the two adjacent links. This approach is suitable for all kinds of ring topology wireless networks whether the base stations are fixed or mobile and can use any kind of polarized antenna.

As it will be shown, the polarizations will be determined by the base stations' locations. Therefore, in a network with fixed base stations, our solution will not require any extra cost compared to networks using horizontal/vertical polarizations. A network with mobile base stations will require a centralized control system and polarization-agile antennas. Such antennas, which polarization state can be changed dynamically, have been studied in [9], [8].

Optimal synthesis of beampattern having any state of polarization using an array of antennas has been addressed in [21], [5]. These polarization synthesis techniques can be used in our model for polarization planning with mobile base stations.

The paper is organized as follows: some general principles of polarization are presented in Section 2. Coverage of a ring topology network with one single channel using polarization is studied in Section 3. Section 4 analyses the implications in terms of spectral efficiency. Section 5 illustrates how the use of polarization proposed in this paper impacts channel allocation. Concluding remarks are given in Section 6.

2 POLARIZATION DIVERSITY

In an electromagnetic wave, electric field and magnetic field are oscillating in two directions orthogonal to the propagation direction and orthogonal to each other. If the fields rotate at the wave frequency, the polarization is circular or elliptical. If the fields oscillate in one single direction, the polarization is linear. By convention, the direction of a linear polarization is the direction of the electric field.

It should be noted that vertical polarization, in the strict sense of the word, is possible only if both base stations are at the same altitude. When antennas were located far away one from each other, a difference in altitude could be neglected relative to the distance and all the nodes of a network could roughly be considered to be co-planar. However, with the deployment of small cells, this difference in altitude will become more and more significant and vertical polarization will often reveal to be impossible.

Polarization diversity enables to reuse the same frequency in a network. In order to avoid interferences, the two polarizations received by a base station at the same frequency must be orthogonal to each other. In practice, due to rotational misalignment of antennas or weather conditions, the signal in one polarization may interfere with the other. However, it is possible to cancel this interference by using the XPIC (Cross-Polarization Interference Cancellation) algorithm [6].

In a chain comprising the nodes A, B and C, the polarization $\mathbf{E}_{AB}$ between A and B may be chosen arbitrarily (upon the condition it is orthogonal to the line AB), but then, the reuse of the same frequency requires that the polarization $\mathbf{E}_{BC}$ between B and C shall be orthogonal to $\mathbf{E}_{AB}$. Since $\mathbf{E}_{BC}$ shall also be orthogonal to the line BC, the direction of $\mathbf{E}_{AB}$ generally determines that of $\mathbf{E}_{BC}$.

This solution can be extended to any number of nodes, provided that all links are LOS: connecting all the nodes of a chain with one single channel is always possible by choosing each polarization orthogonal to the previous one and to the propagation line.

3 POLARIZATION IN RING NETWORKS

Notations: we use bold letters to denote matrices and vectors. $\|\mathbf{u}\|$ denotes the euclidean norm of vector $\mathbf{u}$. $\mathbf{u}\|\mathbf{v}$ means that vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel. The symbol $\times$ denotes the vector product. If $\triangle ABC$ is a triangle, we denote $\angle BAC$ the angle in A.

We assume in the following that all the transmissions are LOS and that the nodes are located far away enough one from each other, so that the signal received by a node does not cause interference to other nodes.

Since the polarization of a link generally determines the polarization of the next one, the use of one single frequency by polarization diversity in a ring topology network faces the question of closing the loop: the first polarization will determine the second one, which will determine the third one and so forth until the last one. The last polarization, which is determined by a sequence of constraints, is not necessarily orthogonal to the first one. In this section, we will study under what conditions it is possible to choose the first polarization in such a way that this orthogonality property is fulfilled.

3.1 Formulating the problem

Let us consider n nodes A_1, A_2,..., A_n. We define the following unitary vectors:

$$\mathbf{u_1} = \frac{\mathbf{A_1A_2}}{\|\mathbf{A_1A_2}\|}, \mathbf{u_2} = \frac{\mathbf{A_2A_3}}{\|\mathbf{A_2A_3}\|}, \ldots, \mathbf{u_n} = \frac{\mathbf{A_nA_1}}{\|\mathbf{A_nA_1}\|}$$

and the polarizations: $\mathbf{E_1}$ between A_1 and A_2, $\mathbf{E_2}$ between A_2 and A_3,..., $\mathbf{E_n}$ between A_n and A_1 (see Fig. 1).

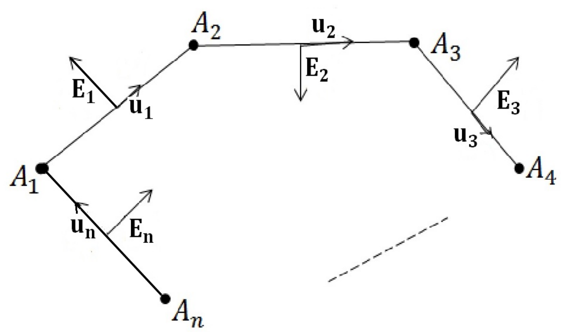

Fig. 1 Ring network.

Since each polarization shall be orthogonal to the previous one and to the propagation direction, it must be parallel to the vector product of these two vectors:

$\mathbf{E_1} || \mathbf{E_n} \times \mathbf{u_1}$
$\mathbf{E_2} || \mathbf{E_1} \times \mathbf{u_2}$
$\vdots$
$\mathbf{E_n} || \mathbf{E_{n-1}} \times \mathbf{u_n}$

Therefore, $\mathbf{E_n}$ must fulfill the following condition:

$$\mathbf{E_n} || ((\mathbf{E_n} \times \mathbf{u_1}) \times \cdots \times \mathbf{u_{n-1}}) \times \mathbf{u_n} \qquad (1)$$

Let us define n vector planes: P_1 orthogonal to $\mathbf{u_1}$, P_2 orthogonal to $\mathbf{u_2}$,..., P_n orthogonal to $\mathbf{u_n}$. We can now define the following n homomorphisms (for simplicity of notation, we put $P_0 = P_n$):

$$\phi_i : P_{i-1} \to P_i$$

$$\mathbf{x} \mapsto \mathbf{x} \times \mathbf{u_i}$$

Then, $\phi = \phi_n \phi_{n-1} \dots \phi_1$ is an endomorphism in P_n.

The above condition (1) can be written:

$$\mathbf{E_n} || \phi(\mathbf{E_n}) \qquad (2)$$

Therefore, the problem can be expressed as the search of an eigenvector for the endomorphism ϕ.

3.2 Matrix expression

3.2.1 Condition for the existence of an eigenvector

It is well known that the eigenvalues of an endomorphism ϕ in a vector plane are the solutions of the equation:

$$X^2 - tr(\phi)X + det(\phi) = 0 \qquad (3)$$

where $tr(\phi)$ and $det(\phi)$ are the trace and the determinant of ϕ, respectively.

This equation has solutions if and only if:

$$tr(\phi)^2 - 4det(\phi) \geq 0 \qquad (4)$$

3.2.2 Choosing an appropriate basis for each plane

For each vector plane P_i, we define two vectors $\mathbf{v_i}$ and $\mathbf{w_i}$ fulfilling the following conditions:

- $\mathbf{v_i}$ is in the plane containing A_i, A_{i+1} and A_{i+2} (for simplicity of notation, we put $A_{n+1} = A_1$ and $A_{n+2} = A_2$).
- ($\mathbf{u_i}$,$\mathbf{v_i}$,$\mathbf{w_i}$) is a direct orthonormal basis of the space.

Note that the choice of $\mathbf{v_i}$ is not unique: two opposite vectors fulfill the condition. Anyone of both can be chosen arbitrarily.

We also define in P_i two vectors $\mathbf{v'_i}$ and $\mathbf{w'_i}$ fulfilling the following conditions:

- $\mathbf{w'_i} = \mathbf{w_{i-1}}$ (for simplicity of notation, we put $\mathbf{w_0} = \mathbf{w_n}$).
- ($\mathbf{u_i}$,$\mathbf{v'_i}$,$\mathbf{w'_i}$) is a direct orthonormal basis of the space.
  The plane containing A_{i-1}, A_i and A_{i+1} is represented in Fig. 2 with the bases ($\mathbf{u_{i-1}}$,$\mathbf{v_{i-1}}$,$\mathbf{w_{i-1}}$) and ($\mathbf{u_i}$,$\mathbf{v'_i}$,$\mathbf{w'_i}$).

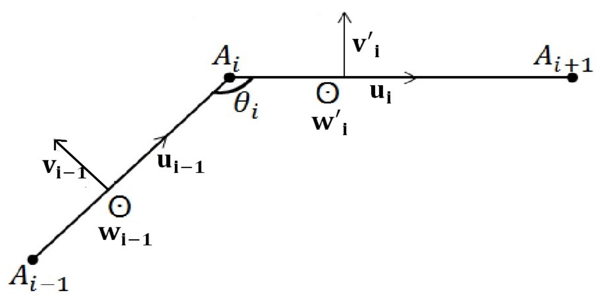

Fig. 2 Vector basis.

3.2.3 Calculation of the matrix

B_i=($\mathbf{v_i}$,$\mathbf{w_i}$) and B'_i=($\mathbf{v'_i}$,$\mathbf{w'_i}$) are two orthonormal bases of P_i. Since ($\mathbf{u_i}$,$\mathbf{v_i}$,$\mathbf{w_i}$) and ($\mathbf{u_i}$,$\mathbf{v'_i}$,$\mathbf{w'_i}$) are both direct orthonormal bases of the space, the change of basis matrix from B_i to B'_i is a rotation matrix:

$$\mathbf{R_i} = \begin{pmatrix} \cos\alpha_i & -\sin\alpha_i \\ \sin\alpha_i & \cos\alpha_i \end{pmatrix} \tag{5}$$

where α_i is the rotation angle of matrix $\mathbf{R_i}$.

We define the angle $\theta_i = \angle A_{i-1}A_iA_{i+1}$.

Since $\mathbf{v_{i-1}} \times \mathbf{u_i} = \sin(\theta_i + \frac{\pi}{2})\mathbf{w'_i} = \cos\theta_i \mathbf{w'_i}$ and $\mathbf{w_{i-1}} \times \mathbf{u_i} = \mathbf{v'_i}$, the matrix of ϕ_i with respect to the bases B_{i-1} and B'_i is:

$$\mathbf{M_i} = \begin{pmatrix} 0 & 1 \\ \cos\theta_i & 0 \end{pmatrix} \tag{6}$$

Therefore, the matrix of ϕ in B_n is:

$$\mathbf{M} = \mathbf{R_n M_n R_{n-1} M_{n-1} \dots R_2 M_2 R_1 M_1} \tag{7}$$

3.3 Solution of the problem

3.3.1 All the nodes are in the same plane

If all the nodes are in the same plane, it is possible to choose B_i and B'_i so that $B_i = B'_i$ for all i. Then, all the rotation matrices equal the identity matrix. Equation (7) becomes:

$$\mathbf{M} = \mathbf{M_n M_{n-1} \dots M_2 M_1} \tag{8}$$

First case: the number of nodes is even
If n is an even number, equations (6) and (8) give:

$$\mathbf{M} = \begin{pmatrix} \cos\theta_1 \cos\theta_3 \dots \cos\theta_{n-1} & 0 \\ 0 & \cos\theta_2 \cos\theta_4 \dots \cos\theta_n \end{pmatrix} \tag{9}$$

In this case, the eigenvectors are obviously $\mathbf{v_n}$ and $\mathbf{w_n}$. This matches the intuitive solution of choosing polarization in the plane containing all the nodes and orthogonal to this plane, alternately.

In addition, in the particular case where $\cos\theta_1 \cos\theta_3 \dots \cos\theta_{n-1} = \cos\theta_2 \cos\theta_4 \dots \cos\theta_n$, then ϕ is a homothety. Any vector is an eigenvector and the first polarization can be chosen arbitrarily.

Second case: the number of nodes is odd
If n is an odd number, equations (6) and (8) give:

$$\mathbf{M} = \begin{pmatrix} 0 & \cos\theta_2 \cos\theta_4 \dots \cos\theta_{n-1} \\ \cos\theta_1 \cos\theta_3 \dots \cos\theta_n & 0 \end{pmatrix} \tag{10}$$

Then, $tr(\phi) = tr(M) = 0$,

and $det(\phi) = det(\mathbf{M}) = -\cos\theta_1 \cos\theta_2 \dots \cos\theta_n$.

Equation (3) becomes:

$$X^2 - \cos\theta_1 \cos\theta_2 \dots \cos\theta_n = 0 \tag{11}$$

This equation has a solution if and only if $\cos\theta_1 \cos\theta_2 \dots \cos\theta_n \geq 0$, which means if and only if the number or obtuse angles in the polygon is even.

Since the total number of angles in the polygon is odd, we can express a necessary and sufficient condition:

Equation (3) has a solution if and only if the number of acute angles in the polygon is odd.

The eigenvectors coordinates in B_n can be easily calculated:

$$\mathbf{E_n} = \lambda \begin{pmatrix} \sqrt{|\cos\theta_2 \cos\theta_4 \ldots \cos\theta_{n-1}|} \\ \pm\sqrt{|\cos\theta_1 \cos\theta_3 \ldots \cos\theta_n|} \end{pmatrix}, \lambda \in \mathbb{R} \tag{12}$$

3.3.2 General case

If we do not assume that all the nodes are in the same plane, the calculation of $\mathbf{M}$ from equation (3) is much more complicated and $tr(\phi)$ cannot be calculated easily. However since the determinants of rotation matrices $\mathbf{R_i}$ equal to 1, we can still calculate $det(\phi)$:

$$det(\phi) = det(\mathbf{M}) = (-1)^n \cos\theta_1 \cos\theta_2 \ldots \cos\theta_n \tag{13}$$

It is therefore possible to express a sufficient, though not necessary, condition for the existence of a solution to equation (3):

If the number of acute angles in the polygon is odd, then $det(\phi) \leq 0$ and equation (3) has a solution.

3.3.3 Particular case of 4 nodes

Theorem 1 *If $n = 4$, equation (3) has always a solution.*

Proof Without loss of generality, we may assume that the nodes have the following coordinates:

$$A_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, A_2 \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}, A_3 \begin{pmatrix} c \\ d \\ e \end{pmatrix}, A_4 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

According to the notations above,

$$\mathbf{u_4} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v_4} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{w_4} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We obtain, after calculation:

$$(((\mathbf{v_4} \times \mathbf{A_1A_2}) \times \mathbf{A_2A_3}) \times \mathbf{A_3A_4}) \times \mathbf{A_4A_1} = (a-1) \begin{pmatrix} 0 \\ d(b-d) + c(a-c) \\ e(b-d) \end{pmatrix} \tag{14}$$

and

$$(((\mathbf{w_4}\times\mathbf{A_1A_2})\times\mathbf{A_2A_3})\times\mathbf{A_3A_4})\times\mathbf{A_4A_1} = \begin{pmatrix} 0 \\ (1-a)ed+bec \\ (1-a)(e^2+c(c-a))+bc(b-d) \end{pmatrix} \quad (15)$$

Therefore, $M = \frac{1}{\|\mathbf{A_1A_2}\|\|\mathbf{A_2A_3}\|\|\mathbf{A_3A_4}\|\|\mathbf{A_4A_1}\|}A$, with

$$A = \begin{pmatrix} (a-1)(d(b-d)+c(a-c)) & (1-a)ed+bec \\ (a-1)e(b-d) & (1-a)(e^2+c(c-a))+bc(b-d) \end{pmatrix} \quad (16)$$

$$tr(A)^2-4det(A) = ((a-1)(d(b-d)+e^2)-bc(b-d))^2+4e^2(a-1)(b-d)((1-a)d+bc) \quad (17)$$

$$tr(A)^2-4det(A) = \big((a-1)(d(b-d)-e^2)-bc(b-d)\big)^2 \geq 0 \quad (18)$$

Therefore, A has eigenvectors and ϕ has eigenvectors. This proves that it is possible to use one single frequency in any ring topology network including four nodes.

3.3.4 Summarized results

The results obtained above are summarized in TABLE 1.

Nodes	**In the same plane**	**Not in the same plane**
Odd number	Solution if and only if there is an odd number of acute angles.	If the number of acute angles is odd, then there is a solution.
Even number≥ 6	There is a solution.	
4 nodes	There is a solution.	

Table 1 One single frequency in a ring network.

3.4 Special cases

Below are three application cases for the results we obtained.

3.4.1 Ring topology with a right angle

If there is a right angle in the ring, e.g. in A_i, then $det(M_i) = 0$ and therefore $det(\phi) = 0$. Equation (3) has a solution.

One can build a solution by choosing $\mathbf{E_i} = \mathbf{u_{i-1}}$ and then applying successively ϕ_{i+1},..., ϕ_n, ϕ_1,...,ϕ_{i-1}. The last vector will be in P_{i-1} and therefore will be orthogonal to $\mathbf{E_i}$.

3.4.2 Triangle topology

As a result of the "odd number of acute angles" condition, equation (3) has a solution if and only if the three angles of the triangle are acute.

The case of the triangle is the best illustration of the advantages of the solution proposed in this paper over the present state of the art: if B is the available bandwidth, the use of horizontal/vertical polarization enables to double the bandwidth, and since the same frequency cannot be used on two adjacent links with the same type of polarization, it is possible to allocate up to $\frac{2}{3}B$ to each link. By defining three polarizations orthogonal to each other, our solution enables to allocate B to each link.

As an example, the polarizations in the case of an equilateral triangle are given below (see Fig. 3).

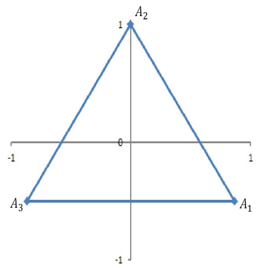

Fig. 3 Equilateral triangle.

With the notations above, we obtain the following polarizations:

$$\mathbf{E_1} = \begin{pmatrix} -1/\sqrt{2} \\ -1/\sqrt{6} \\ \pm 1/\sqrt{3} \end{pmatrix} \mathbf{E_2} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{6} \\ \pm 1/\sqrt{3} \end{pmatrix} \mathbf{E_3} = \begin{pmatrix} 0 \\ \sqrt{2/3} \\ \pm 1/\sqrt{3} \end{pmatrix}$$

3.4.3 Regular pentagon topology

In a convex regular pentagon, all angles are obtuse. The "odd number of acute angles" condition is not fulfilled and polarization diversity does not enable the use of one single frequency over the ring. However, if the pentagon vertices are connected according to a sheriff star, then all the angles are acute and the use of one single frequency is possible (see Fig. 4).

The method described above enables to calculate the polarizations:

$$\mathbf{E_1} = \begin{pmatrix} \pm 0.437 \\ \mp 0.602 \\ 0.669 \end{pmatrix} \mathbf{E_2} = \begin{pmatrix} -0.707 \\ 0.230 \\ \pm 0.669 \end{pmatrix} \mathbf{E_3} = \begin{pmatrix} \pm 0.707 \\ \pm 0.230 \\ 0.669 \end{pmatrix}$$

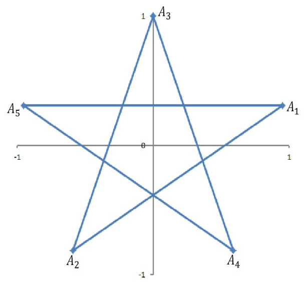

Fig. 4 Sheriff star.

$$\mathbf{E_4} = \begin{pmatrix} -0.437 \\ -0.602 \\ \pm 0.669 \end{pmatrix} \mathbf{E_5} = \begin{pmatrix} 0 \\ 0.743 \\ \pm 0.669 \end{pmatrix}$$

Another solution also enables the use of one single frequency: a pentagon with three long sides and two short sides (see Fig. 5).

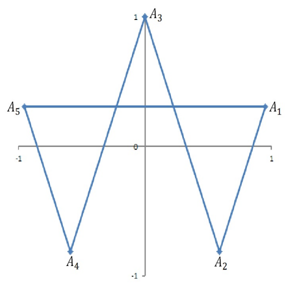

Fig. 5 Pentagon with 3 long sides and 2 short sides.

The method described above enables to calculate the polarizations:

$$\mathbf{E_1} = \begin{pmatrix} \pm 0.309 \\ \mp 0.1 \\ 0.292 \end{pmatrix} \mathbf{E_2} = \begin{pmatrix} -0.278 \\ -0.09 \\ \pm 0.263 \end{pmatrix} \mathbf{E_3} = \begin{pmatrix} \pm 0.25 \\ \mp 0.081 \\ 0.236 \end{pmatrix}$$

$$\mathbf{E_4} = \begin{pmatrix} -0.225 \\ -0.073 \\ \pm 0.213 \end{pmatrix} \mathbf{E_5} = \begin{pmatrix} 0 \\ 0.946 \\ \pm 0.325 \end{pmatrix}$$

In a pentagon, the use of horizontal/vertical polarization enables to allocate at best 80% of the available bandwidth to each link. An example of optimal allocation can be obtained by dividing the available band into 5 equal bands B_1, B_2, B_3, B_4, B_5 and performing the allocation shown on TABLE 2.

Link	**Frequency bands**	**Polarizations**
A_1A_2	B_1, B_4	Horizontal/Vertical
A_2A_3	B_2, B_5	Horizontal/Vertical
A_3A_4	B_3, B_1	Horizontal/Vertical
A_4A_5	B_4, B_2	Horizontal/Vertical
A_5A_1	B_5, B_3	Horizontal/Vertical

Table 2 Frequency and polarization allocation in a pentagon.

In comparison, our solution enables to allocate the whole bandwidth to each link.

More generally, for any odd number n, in an n-sided polygon, the use of horizontal/vertical polarization enables to allocate at best $\frac{n-1}{n}B$ to each link, B being the available bandwidth. When our solution is applicable, it enables to allocate the whole bandwidth to each link.

In an even-number sided polygon, the use of horizontal/vertical polarization also enables to allocate the whole bandwidth to each link by sharing the band into two halves and allocating alternately to each link the lower part and the upper part, always with two polarizations. However, since the frequencies in the neighborhood of the band bounds are often disturbed in practice by interferences from the contiguous band, our solution should be preferable though the advantage is less evident than that of the odd-number sided polygon case.

4 Spectral efficiency

The benefits of inclined polarization can be highlighted by its consequences in terms of spectral efficiency. We will hereafter compare the performances of polarization multiplexing and inclined polarization in the two ring topologies described above: triangle and regular pentagon.

According to the Shannon-Hartley theorem, the maximum bit rate which can be transmitted over a channel is given by the expression:

$$C = B \log_2(1 + \frac{S}{N}) \tag{19}$$

where

C is the channel capacity in bit/s;

B is the channel's bandwidth in Hertz;

S is the average signal power over the bandwidth, in Watt;

N is the average noise or interference power over the bandwidth, in Watt.

Classical polarization multiplexing, such as horizontal/vertical, enables frequency reuse. On the other hand, it creates some cross-polarization interference which will not be totally suppressed by the XPIC algorithm. The ultra-high-performance antenna discrimination between two orthogonal polarizations is typically close to $40\,dB$ [11]. We will retain in the following a cross-polarization interference of $-30\,dB$. Therefore, an interference noise of λS will be added to the thermal noise N, with $\lambda = 10^{-3}$.

By comparison to classical polarization multiplexing, inclined polarization enhances the frequency reuse possibilities for the same cross-polarization interference. However, if a change in the network topology is required, inclined polarization will cause higher distances and therefore lower received power. Therefore, the spectral efficiency comparison for the regular pentagon depends on the propagation model.

In the following, we will assume that the path attenuation is proportional to d^α, d being the distance and α a real number greater or equal to 2. Since all the transmissions are assumed to be LOS, attenuation due to vegetation and obstacles can be ignored. However, attenuation due to gas absorption and precipitation can be added to the free-space loss. Therefore, in order to evaluate the impact of the propagation model on the relevance of our solution, we will perform the comparisons in the scope of two different assumptions:

- $\alpha = 2$ (free-space loss model);
- $\alpha = 3$.

4.1 Triangle topology

4.1.1 No polarization

Since the same frequency cannot be used in two adjacent links, only one third of the bandwidth is available for each link. Therefore, the spectral efficiency is:

$$\frac{C}{B} = \frac{1}{3}\log_2(1 + \frac{S}{N}) \tag{20}$$

4.1.2 Polarization multiplexing

Polarization multiplexing doubles the bandwidth available for each link. Therefore:

$$\frac{C}{B} = \frac{2}{3}\log_2(1 + \frac{S}{N + \lambda S}) = \frac{2}{3}\log_2(1 + \frac{S/N}{1 + \lambda S/N}) \tag{21}$$

4.1.3 Inclined polarization

Inclined polarization enables the use of all the available bandwidth:

$$\frac{C}{B} = \log_2(1 + \frac{S}{N + \lambda S}) = \log_2(1 + \frac{S/N}{1 + \lambda S/N}) \tag{22}$$

As a result of Equations (20), (21) and (22), for a given signal-to-noise ratio, the spectral efficiency is almost doubled with horizontal/vertical polarization as long as the dominant noise is the thermal noise. When the signal-to-noise ratio tends to infinity, the use of polarization is not relevant anymore.

As shown on Fig. 6, inclined polarization improves by 50% the spectral efficiency offered by horizontal/vertical polarization. This improvement does not depend upon the propagation model.

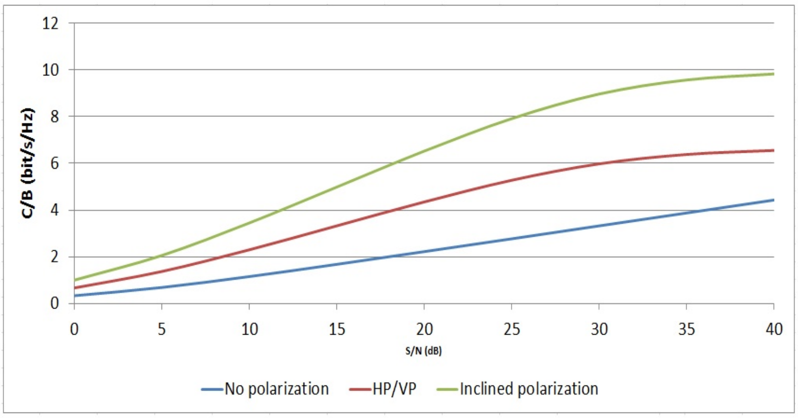

Fig. 6 Spectral efficiency in a triangle.

4.2 Regular pentagon topology

In order to compare the spectral efficiencies of the various strategies, we will assume that the power emitted by each node is such that the received power at the adjacent node of the convex pentagon is S.

In a pentagon, the best frequency allocation strategy is given in TABLE 2. It enables to allocate two fifths of the total bandwidth without polarization and four fifths of the total bandwidth with polarization multiplexing.

4.2.1 No polarization

$$\frac{C}{B} = \frac{2}{5}\log_2(1 + \frac{S}{N}) \tag{23}$$

4.2.2 Polarization multiplexing

$$\frac{C}{B} = \frac{4}{5}\log_2(1+\frac{S}{N+\lambda S}) = \frac{4}{5}\log_2(1+\frac{S/N}{1+\lambda S/N}) \tag{24}$$

4.2.3 Inclined polarization - Sheriff star

Taking the notations of Fig. 4, we first calculate the ratio between the side of a sheriff star and the side of a regular pentagon:

$$\frac{A_1A_2}{A_1A_4} = \frac{\sin\frac{3\pi}{5}}{\sin\frac{\pi}{5}} = 1 + 2\cos\frac{2\pi}{5} = \frac{\sqrt{5}+1}{2} \tag{25}$$

Therefore, for the same emitting power, the ratio between the received power in a regular pentagon and the received power in a sheriff star is:

$$\frac{{A_1A_2}^{\alpha}}{{A_1A_4}^{\alpha}} = \left(\frac{\sqrt{5}+1}{2}\right)^{\alpha} \tag{26}$$

As a result, the spectral efficiency is:

$$\frac{C}{B} = \log_2\left(1 + \frac{\left(\frac{2}{\sqrt{5}+1}\right)^{\alpha} S/N}{1+\lambda\left(\frac{2}{\sqrt{5}+1}\right)^{\alpha} S/N}\right) \tag{27}$$

– If $\alpha = 2$ and $\lambda = 10^{-3}$, this equation becomes:

$$\frac{C}{B} = \log_2\left(1 + \frac{\frac{2}{\sqrt{5}+3} S/N}{1+10^{-3}\frac{2}{\sqrt{5}+3} S/N}\right) \tag{28}$$

– If $\alpha = 3$ and $\lambda = 10^{-3}$, this equation becomes:

$$\frac{C}{B} = \log_2\left(1 + \frac{\frac{1}{\sqrt{5}+2} S/N}{1+10^{-3}\frac{1}{\sqrt{5}+2} S/N}\right) \tag{29}$$

4.2.4 Inclined polarization - Pentagon with 3 long sides and 2 short sides

In order to obtain a given spectral efficiency $\frac{C}{B}$, the required signal-to-noise ratio is different for a long side and for a short side (see Fig. 5):

Long side:

$$\left(\frac{S}{N}\right)_{long} = \left(\frac{\sqrt{5}+1}{2}\right)^{\alpha} \frac{2^{\frac{C}{B}}-1}{1+\lambda-\lambda 2^{\frac{C}{B}}} \tag{30}$$

Short side:

$$\left(\frac{S}{N}\right)_{short} = \frac{2^{\frac{C}{B}}-1}{1+\lambda-\lambda 2^{\frac{C}{B}}} \tag{31}$$

Therefore, the average signal-to-noise ratio is:

$$\left(\frac{S}{N}\right)_{average} = \frac{1}{5}\left(3\left(\frac{S}{N}\right)_{long} + 2\left(\frac{S}{N}\right)_{short}\right) \tag{32}$$

$$\left(\frac{S}{N}\right)_{average} = \frac{1}{5}\left(3\left(\frac{\sqrt{5}+1}{2}\right)^{\alpha} + 2\right)\frac{2^{\frac{C}{B}} - 1}{1 + \lambda - \lambda 2^{\frac{C}{B}}} \tag{33}$$

The spectral efficiency can now be expressed as a function of the average signal-to-noise ratio:

$$\frac{C}{B} = \log_2\left(1 + \frac{\frac{5}{3\left(\frac{\sqrt{5}+1}{2}\right)^{\alpha}+2}\left(\frac{S}{N}\right)_{average}}{1 + \lambda\frac{5}{3\left(\frac{\sqrt{5}+1}{2}\right)^{\alpha}+2}\left(\frac{S}{N}\right)_{average}}\right) \tag{34}$$

– If $\alpha = 2$ and $\lambda = 10^{-3}$, this equation becomes:

$$\frac{C}{B} = \log_2\left(1 + \frac{\frac{10}{3\sqrt{5}+13}\left(\frac{S}{N}\right)_{average}}{1 + 10^{-3}\frac{10}{3\sqrt{5}+13}\left(\frac{S}{N}\right)_{average}}\right) \tag{35}$$

– If $\alpha = 3$ and $\lambda = 10^{-3}$, this equation becomes:

$$\frac{C}{B} = \log_2\left(1 + \frac{\frac{5}{3\sqrt{5}+8}\left(\frac{S}{N}\right)_{average}}{1 + 10^{-3}\frac{5}{3\sqrt{5}+8}\left(\frac{S}{N}\right)_{average}}\right) \tag{36}$$

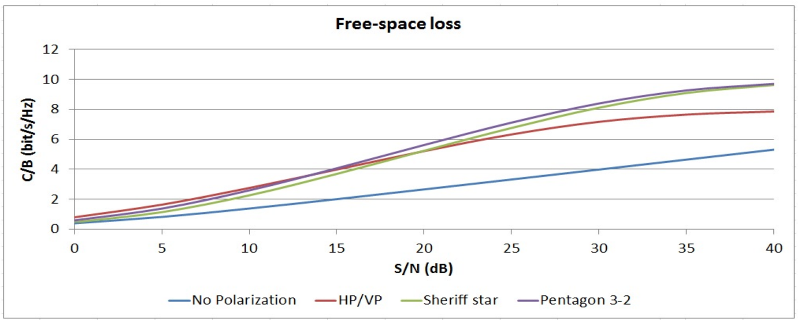

Fig. 7 Spectral efficiency in a pentagon - Free-Space Loss.

As shown on Fig. 7, for $\alpha = 2$, the sheriff star is a more relevant solution than the horizontal/vertical polarization if the signal-to-noise ratio is greater than $19.7\,dB$. The pentagon with three long sides and two short sides is the best solution if the signal-to-noise ratio is greater than $13.5\,dB$.

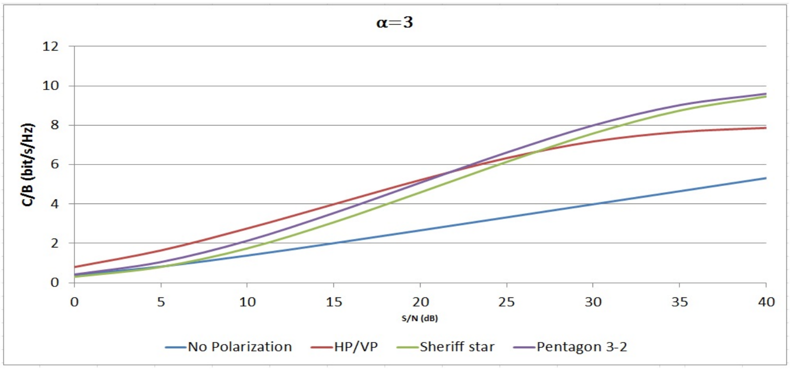

Fig. 8 Spectral efficiency in a pentagon - $\alpha = 3$.

As shown on Fig. 8, for $\alpha = 3$, the sheriff star is a more relevant solution than the horizontal/vertical polarization if the signal-to-noise ratio is greater than $26.8\,dB$. The pentagon with three long sides and two short sides is the best solution if the signal-to-noise ratio is greater than $21.8\,dB$.

In any case, the pentagon with three long sides and two short sides always offers a better spectral efficiency than the sheriff star.

5 Channel Allocation

Channel allocation in a wireless network is usually considered as a matter of graph coloring. To each channel corresponds a color. Allocating channels in a network while avoiding channel interferences is equivalent to coloring the edges of a graph while ensuring that two edges having a common vertex are not of the same color. As long as the use of polarization is restricted to HP/VP, this paradigm is not affected: polarization remains a black box which doubles the number of channels and channel allocation is still a matter of graph coloring.

However, the use of polarization we propose in this paper has implications on channel allocation. These implications will be illustrated by the following example.

Let us consider nine nodes A, B, C, A', B', C', A'', B'' and C''. We assume that the triangles ABC, $A'B'C'$, $A''B''C''$, $AA'A''$, $BB'B''$ and $CC'C''$ are connected and that each one of these triangles has three acute angles.

We also assume that all nodes are located far enough away one from each other to avoid any interference between two links which do not share a common node (see Fig. 9).

For the sake of clarity, triangles ABC, $A'B'C'$ and $A''B''C''$ are represented in black, triangle $AA'A''$ is represented in green, triangle $BB'B''$ is represented in purple and triangle $CC'C''$ is represented in orange. These colors are not related to the concept of graph coloring.

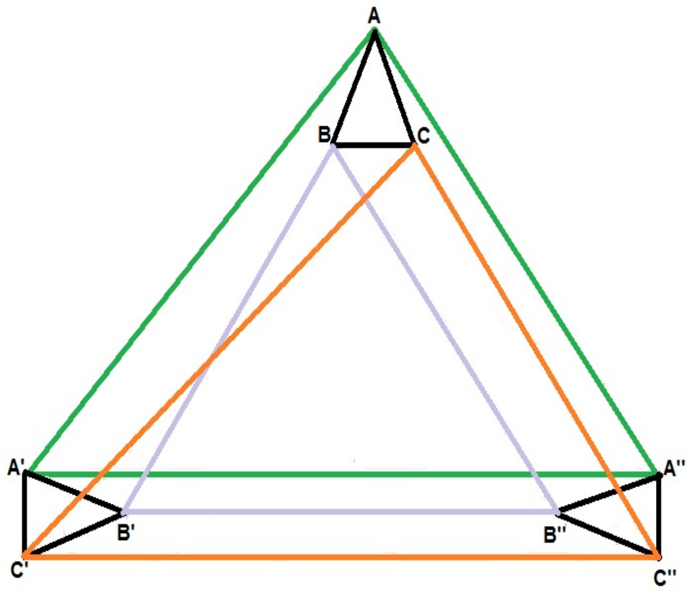

Fig. 9 Example of wireless network.

Theorem 2 *Covering the network of Fig. 9 by a graph coloring approach requires at least 5 channels.*

Proof Let us assume that the network is covered by 4 channels, a, b, c and d. Since each one of the 9 vertices of the network belongs to 4 edges, each one of the 4 channels, a, b, c and d, must be used by exactly one of these 4 edges.

Triangle ABC requires three different channels. Without any loss of generality, we can assume that a is the channel used for the edge BC, b the channel used for the edge AC, and c the channel used for the edge AB.

Then, the two edges AA' and AA'' must use the channels a and d. As well, the two edges BB' and BB'' must use the channels b and d and the two edges CC' and CC'' must use the channels c and d.

Therefore, channel d must be used at least twice between triangle ABC and one of the triangles $A'B'C'$ or $A''B''C''$. Without any loss of generality, we can assume that channel d is used for the edges AA'' and BB''. As a result, channel a is used for the edge AA' and channel b is used for the edge BB'.

If channel d is allocated to the edge CC', then no edge containing C'' can use channel d: $C''C$ and $C''C'$ because channel d is already used for $C'C''$, and $C''A''$ (resp. $C''B''$) because channel d is already used for AA'' (resp. BB''). This contradicts the fact that each one of the four channels must be used by exactly one of the four edges containing C''. Therefore, it is necessary to allocate channel c to CC' and channel d to CC''.

Then, since channel d is already allocated to AA'', it cannot be allocated to any of the edges AA' or $A'A''$. Therefore, it must be allocated to one of the two edges $A'B'$ or $A'C'$.

Likewise, channel d must be allocated to one of the two edges $B'A'$ or $B'C'$, and one of the two edges $C'A'$ or $C'B'$. This implies it must be used twice in the triangle $A'B'C'$. This is impossible.

Therefore at least 5 channels (and therefore at least 3 frequencies if polarization diversity is restricted to HP/VP) are required in order to cover the network. Fig. 10 provides an example of such a covering.

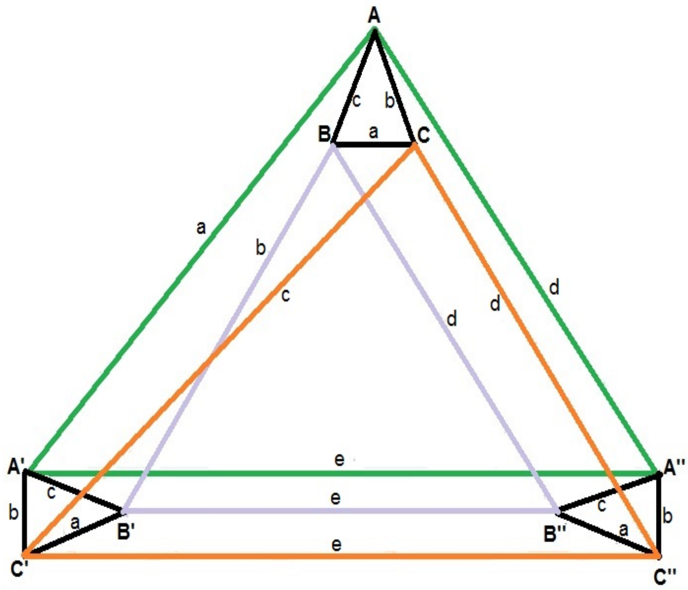

Fig. 10 Channel allocation in a wireless network.

By comparison, the use of inclined polarization enables network coverage with 2 frequencies only: one frequency for the triangles ABC, $A'B'C'$ and $A''B''C''$, and another frequency for the triangles $AA'A''$, $BB'B''$ and $CC'C''$.

6 Conclusion

Polarization diversity enables significant improvements regarding frequency allocation in various ring topology networks. In this paper, we show that the appropriate polarizations are the eigenvectors of an endomorphism. We give a necessary and sufficient condition for the existence of such a solution in the case all the base stations are on the same plane and a sufficient condition in the general case. We show that the appropriate polarization on a given link of a ring depends upon the position of all the nodes of the ring. The choice of polarization is not restricted to vertical and horizontal: inclined polarization can be an efficient mean to improve spectrum efficiency, even when all the base stations of a network are on a horizontal plane. It is always the best option when compatible with the network topology and can be considered as an alternative even if a change in the network topology is required. As shown in the examples above, inclined polarization is particularly efficient in triangles, where it can improve the spectrum efficiency by 50% in comparison with horizontal/vertical polarization and it is still fairly efficient in a number of other configurations. Furthermore, polarization diversity brings a new paradigm regarding channel allocation in a wireless network. Beyond being a black box which doubles the number of available channels, polarization diversity enables much more significant improvements on resource use. Further work still has to be done regarding topology optimization and channel allocation taking into account the opportunities inclined polarization can offer as described in the present paper.

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