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Fixed Point Theorem and its Limitation for Derivation of Wi-Fi Networks Performance

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***Abstract*—The performance analysis of the distributed access mechanism of the IEEE 802.11 standard also known as Wi-Fi has been extensively studied for more than a decade. Several mathematical models that describe this mechanism have been developed. Some of them using Markov chain and others based on geometrical distribution. One important point often omitted is the way to solve the problem numerically. Indeed, the set of equations exhibited by the model is very complex to be solved. A natural way to solve a set of nonlinear equations is to use the fixed point theorem. In this paper, we present an application of the fixed point theorem method to solve the problem of throughput derivation. After an involved series of mathematical derivations, we show why this method is limited since it requires extreme condition on the error probability.**

I. Introduction

Since the first IEEE 802.11 standard (a.k.a Wi-Fi) in 1997 [1], researchers kept on trying to model the behavior of access mechanisms to the wireless medium. Indeed, it is difficult to model the distributed coordination function (DCF), particularly due to the number of parameters that change during the transmission. A relevant and efficient model would constitute a key to assist the deployment of wireless networks, which is currently done in a quasi empirical way. We distinguish two main categories of models. Since 1996, the CSMA/CA mechanism used by the DCF and its performances were studied by Bianchi [2]. This model which is based on **Markov chains** was published in 1998 [3]-[4]. In parallel, Cali et al. [5] developed a model based on **geometric distributions**.The so-called Bianchi model is based on a two dimensional Markov chain. The first dimension $s(t)$ indicates the backoff stage which represents the number of transmission attempts which failed. The second dimension $b(t)$, indicates the value of the backoff timer, which corresponds to the number of time slots to wait before being able re-initialize a transmission after a failure. This model is a good fit for a saturated medium because it assumes that the station always has data to transmit. This implies that the results represent the maximum throughput offered by a WiFi cell. Cali's model also allows to compute the maximum flow offered in saturated mode for the DCF but in his model the backoff time is evaluated as the average of a geometric distribution. Although both Cali's and Bianchi's models use the same approximations,a major difference between those two models lies in the way of computing the probability for a station transmitting at a time t (the computation being easier in Cali's model).

The IEEE 802.11e standard [6] including mechanisms for QoS management, has also been studied, in saturated mode [7], [8], [9]. These models are both based on Markov chains and extend the Bianchi's model. They assume that the system is in saturated mode and that the channel is ideal. Our model proposed in [10] is an improvement of the IEEE 802.11 and IEEE 802.11e existing models, and we chose to follow the methodology presented in [7]. Indeed, our objective is to provide a more realistic and extensive model. Thus, we suggest the following improvements: we consider a non-ideal channel (i.e., which introduces errors into the packets, according to a fixed error probability), and we consider that stations may be idle (i.e. the emission buffer of the network card can be empty).

In the following section (section II) we present briefly the basics of our model that can be found in more details in [10]. Then in section III we detail the way we apply the fixed point theorem method to solve the set of equations that arises from our model. We conclude in section IV

II. Notations and Model

As mentioned, our model is based on Markov chain with a simplified representation in figure 1. This models the behavior of an access category (AC) managed by EDCA, for a given station. In order to simplify the diagram, we did not represent all the transition probabilities from one state to another. Our model comprises a great number of indices and variables, which are summarized in Table I. In our model, we introduced (relatively to our Bianchi's model mentioned in section I) a fourth dimension, denoted by $e(t)$, such that $e(t) = 1$ if the transmission is corrupted but did not undergo a collision and $e(t) = 0$ otherwise. This variable was introduced in order to distinguish between a transmission failed because of a collision and that which fails because of an error. Let P_i be the collision probability and P_b the probability that the channel is busy. At time t, a state of a given AC_i is fully determined by the quadruplet (j, k, d, e) which corresponds to the values taken respectively by each dimension.

Let $b_{j,k,d,e}$ be the probability to be in state (j,k,d,e), when the system is steady (in other words when $t \to +\infty$). P_{fi} stands for the probability of a failed transmission, due to collision or error. In the following, all the probabilities $b_{j,k,d,e}$ have to be indexed to i the index of the access categories AC_i and thus have to be read as b_{j,k,d,e_i}. For purpose of readability we omit the index in further calculations.

We calculated those probabilities using the same methodology as [4] and [7], but it was naturally necessary to adapt the equations and calculations to the requirements and states of our model.

We obtained:

$$b_{j,0,0,0} = (P_{fi})^j \times b_{0,0,0,0} \qquad (1)$$
$$0 \le j \le m+h$$

$$\begin{aligned} b_{0,k,0,0} &= \binom{W_0-k+1}{1} \\ &\quad \times \frac{1}{W_0+1}\left[(1-P_b)b_{-1,0,0,0} + P_b \sum_{d=0}^{A} b_{-4,0,d,0}\right] \\ &= \frac{W_0-k+1}{W_0+1} \\ &\quad \times \left[\frac{(1-P_b)q}{(1-P_b)} + P_b \sum_{d=0}^{A}(1-q)(1-P_b)^{A-d}\right] \\ &= \frac{W_0-k+1}{W_0+1}\left[q + P_b(1-q)\sum_{d=0}^{A}(1-P_b)^{A-d}\right] \\ &= \frac{W_0-k+1}{W_0+1}\left[q + P_b(1-q)\frac{1-(1-P_b)^{A+1}}{P_b}\right] \\ &= \frac{W_0-k+1}{W_0+1} \\ &\quad \times (1-(1-q)(1-P_b)^{A_i+1}) \\ &\quad \times b_{0,0,0,0} \\ &\quad 1 \le k \le W_j \end{aligned} \qquad (2)$$

III. Application of the Fixed Point Theorem

Theorem 1. *Let $X \subset \mathbb{R}^n$, $n \ge 1$ a closed set, and $\|\cdot\|$ a norm on $\mathbb{R}$. Let $F : X \to X$ be a function such that there is $\alpha \in [0,1[$ such that*

$$\|F(x) - F(y)\| \le \alpha\|x-y\|$$

for all $x, y \in X$. Thus there is a single point $P \in X$ such that $F(P) = P$. Moreover, let $x_0 \in X$ and let $(x_k)_{k\in\mathbb{N}}$ the sequence defined by induction by $x_{k+1} = F(x_k)$ for $k \in \mathbb{N}$. Therefore $\lim_{k\to+\infty} x_k = P$ and we have

$$\|x_k - P\| \le \alpha^k\|x_0 - P\| \qquad (CV)$$

for all $k \in \mathbb{N}$.

Var.	Explanation
s(t)	Number of retry at time t
b(t)	Backoff timer at time t
υ(t)	Timer in transmission, collision, error or frozen period
e(t)	If error occurs e(t)=1 else 0
j,k,d,e	Value of s(t),b(t),$\upsilon(t)$,e(t) respectively
i	Index of the Access Category ACi i= 0, 1, 2,or 3
A_i	Value of AIFSi decreased by 1
N	Value of the initial frozen timer
W_j	Maximal value of the backoff timer
m	Number of maximum retry with Wj increasing
m + h	Number of maximum retry before discarding the packet
$P_i(P_e)$	collision (error) probability for ACi
P_b	Probability that the channel is busy
q	Probability that the buffer isn't empty
T_e,T_c	time to detect an error,collision
γ,T_s	time for propagation, successful transmission

TABLE I
Variables and constants of the model

Remark: A priori, one does not know P, But thanks to the inequality (CV), we get a very good approximation because the convergence is very fast($\alpha < 1$!!). This is the reason why researchers try as soon as possible to reduce to a fixed point problem .

Step 1: nomenclature.. We denote $\vec{\tau} = (\tau_0, ..., \tau_3)$ et $\vec{b} = (b_0, ..., b_3)$, where for simplicity $b_0, ..., b_3$ denote the vector coordinates of $b_{0,0,0,0}$. We denote

$$\begin{aligned} p(\vec{\tau})_i &= (1-p_e)p_i + p_e \\ &= (1-p_e)(1-(1-\Pi_{j=0}^{3}(1-\tau_j)\Pi_{j=i+1}^{3}(1-\tau_j)) + p_e \end{aligned}$$

We have the relationship

$$\tau_i = (1 + p(\vec{\tau})_i + ... + (p(\vec{\tau})_i)^{m+h})b_i$$

for all i, and thus,

$$b^i = \frac{\tau_i}{1 + p(\vec{\tau})_i + ... + (p(\vec{\tau})_i)^{m+h}}.$$

Then we define the function $f : \mathbb{R}^4 \to \mathbb{R}^4$ with its coordinates:

$$(f(\vec{\tau}))_i = \frac{\tau_i}{1 + p(\vec{\tau})_i + ... + (p(\vec{\tau})_i)^{m+h}},$$

and, for $\vec{b}$ fixed, we try to solve $f(\vec{\tau}) = \vec{b}$, i.e.

$$F(\vec{\tau}) = \vec{\tau},$$

with

$$F(\vec{\tau}) = \vec{\tau} - f(\vec{\tau}) + \vec{b}.$$

It was reduced to a fixed point theorem problem: we will try now to apply the theorem. Let fixed first of all a "suitable" norm:

$$\|x\| = \sup\{|x_i| / i = 0, 1, 2, 3, 4\}.$$

Step 2: Shrinking. Let $\vec{\tau}, \vec{\tau}' \in [0,1]^4$. Let $i \in \{0,1,2,3\}$. We have

$$
\begin{aligned}
&|(F(\vec{\tau}) - F(\vec{\tau}'))_i| \\
&= \Bigg| -(\vec{\tau} - \vec{\tau}')_i \left(1 - \frac{1}{1 + p(\vec{\tau})_i + ... + (p(\vec{\tau})_i)^{m+h}}\right) \\
&\quad - (\vec{\tau}')_i \Bigg(\frac{1}{1 + p(\vec{\tau})_i + ... + (p(\vec{\tau})_i)^{m+h}} \\
&\qquad - \frac{1}{1 + p(\vec{\tau}')_i + ... + (p(\vec{\tau}')_i)^{m+h}}\Bigg)\Bigg| \\
&\leq \left(1 - \frac{1}{m+h+1}\right) \|\vec{\tau} - \vec{\tau}'\| \\
&\quad + \|\vec{\tau}'\| \frac{\sum_{k=1}^{m+h} (p(\vec{\tau})_i)^k - (p(\vec{\tau}')_i)^k}{\left(\sum_{k=0}^{m+h} (p(\vec{\tau})_i)^k\right)\left(\sum_{k=0}^{m+h} (p(\vec{\tau}')_i)^k\right)}
\end{aligned} \tag{3}
$$

The problem is to evaluate the second term of the right side. We put $A = p(\vec{\tau})_i$ and $B = p(\vec{\tau}')_i$. With a small calculation we get:

$$
\begin{aligned}
\sum_{k=1}^{m+h} A^k - B^k &= \sum_{k=1}^{m+h} (A - B) \left(\sum_{j=0}^{k-1} A^j B^{k-1-j}\right) \\
&= (A - B) \sum_{j=0}^{m+h-1} \sum_{k=j}^{m+h-1} A^j B^{k-j} \\
&= (A - B) \sum_{j=0}^{m+h-1} A^j \sum_{k=0}^{m+h-1-j} B^k
\end{aligned}
$$

Because $A, B > 0$, we get

$$
\left|\sum_{k=1}^{m+h} A^k - B^k\right| \leq |A - B| \cdot \left(\sum_{j=0}^{m+h-1} A^j\right) \cdot \left(\sum_{j=0}^{m+h-1} B^j\right)
$$

and thus

$$
\begin{aligned}
&\frac{\left|\sum_{k=1}^{m+h} A^k - B^k\right|}{\left(\sum_{j=0}^{m+h} A^j\right) \cdot \left(\sum_{j=0}^{m+h} B^j\right)} \\
&\leq |A - B| \cdot \left(1 - \frac{A^{m+h}}{\sum_{j=0}^{m+h} A^j}\right) \cdot \left(1 - \frac{B^{m+h}}{\sum_{j=0}^{m+h} B^j}\right)
\end{aligned}
$$

Because $A = p(\vec{\tau})_i \in [p_e, 1]$ and also for B, by taking formula (3), we get

$$
\begin{aligned}
|(F(\vec{\tau}) - F(\vec{\tau}'))_i| &\leq \left(1 - \frac{1}{m+h+1}\right) \|\vec{\tau} - \vec{\tau}'\| \\
&\quad + |p(\vec{\tau})_i - p(\vec{\tau}')_i| \left(1 - \frac{p_e^{m+h}}{m+h+1}\right)^2
\end{aligned} \tag{4}
$$

It remains therefore to evaluate $p(\vec{\tau})_i - p(\vec{\tau}')_i$.

We write it: $p(\vec{\tau})_i = (1 - p_e)\Pi_{j=0}^{3}(1 - \tau_j)^{M_j}$, with $M_j \in \{M - 1, M\}$.

For $k \in \{0, .., 3\}$, we have

$$
\frac{\partial p(\vec{\tau})_i}{\partial \tau_k} = -(1 - p_e) \cdot (1 - \tau_k)^{M_k - 1} \Pi_{j \neq k} (1 - \tau_j)^{M_j}.
$$

We assume henceforth that $M \geq 2$. Then we get

$$
\left|\frac{\partial p(\vec{\tau})_i}{\partial \tau_k}\right| \leq (1 - p_e)M
$$

Thus for $\vec{\tau}, \vec{\tau}'$, there is $t_0 \in]0, 1[$ such that

$$
\begin{aligned}
|p(\vec{\tau})_i - p(\vec{\tau}')_i| &= \left|\sum_{k=0}^{4} \frac{\partial p_i}{\partial \tau_k}(t\vec{\tau} + (1 - t)\vec{\tau}') \cdot (\tau_k - \tau'_k)\right| \\
&\leq 4(1 - p_e)M\|\vec{\tau} - \vec{\tau}'\|
\end{aligned} \tag{5}
$$

If we substitute (5) into (4), we get

$$
|F(\vec{\tau}) - F(\vec{\tau}')| \leq \alpha\|\vec{\tau} - \vec{\tau}'\|
$$

with

$$
\alpha = 1 - \frac{1}{m+h+1} + 4(1 - p_e)M \left(1 - \frac{p_e^{m+h}}{m+h+1}\right)^2. \tag{6}
$$

Thus,if we manage to get p_e close to 1 (i.e. a big probability of error in a packet !), then we have $\alpha < 1$.

remark: The problem is that in the formula of α, if M is big, we are in trouble!

We can maybe refine as following: let be $\delta \in [0, 1]$ and let us consider the domain

$$
D_\delta = \{\vec{\tau} \in [0, 1]^4 / \exists j \text{ tel que } 1 - \tau^j \leq \delta\}
$$

.

Then we get that

$$
\left|\frac{\partial p(\vec{\tau})_i}{\partial \tau_k}\right| \leq (1 - p_e)M\delta^{M-2}
$$

for all $\vec{\tau} \in D_\delta$. In this case, if $\delta < 1$, we have $\lim_{M \to +\infty} M\delta^{M-2} = 0$, and we get back a small coefficient in the formula of α.

Step 3: F is defined on $[0, 1]^4$, and it has to get its value in $[0, 1]^4$ too. Let be $i \in \{0, ..., 3\}$. With the same calculation as in step 2, we get

$$
|F(\tau)_i| \leq \left(1 - \frac{1}{m+h+1}\right) \|\vec{\tau}\| + \|\vec{b}\|
$$

Thus if we set that

$$
\|\vec{b}\| \leq \frac{1}{m+h+1},
$$

we have $|F(\vec{\tau})_i| \leq 1$. The concern is that $F(\vec{\tau})_i$ is not always positive... Thus we set

$$
\tilde{F}(\vec{\tau}) = (\max\{0, F(\vec{\tau})_0\}, ..., \max\{0, F(\vec{\tau})_3\})
$$

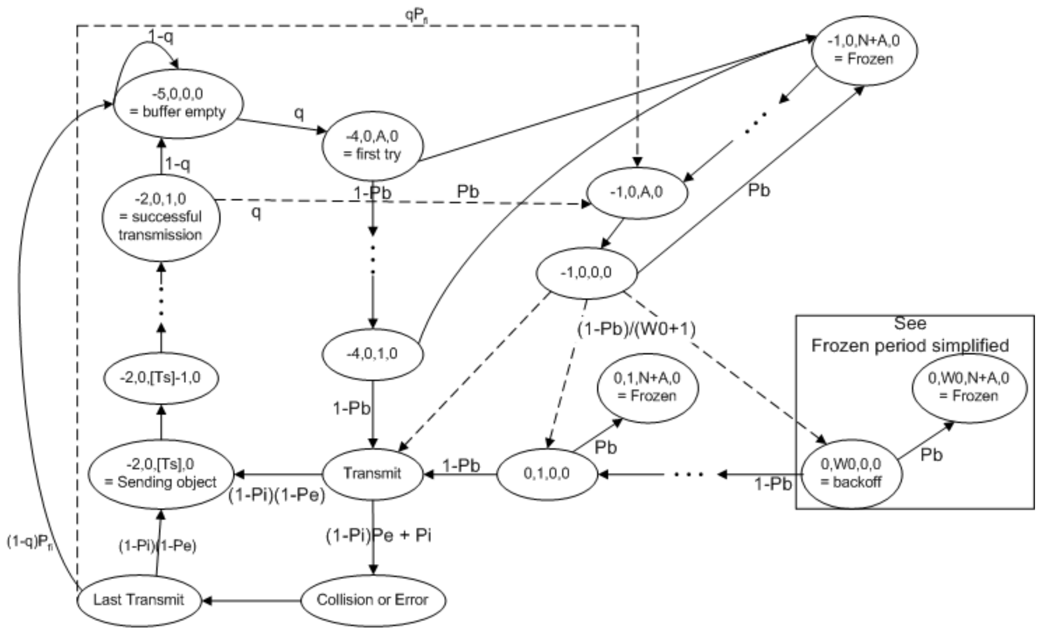

Fig. 1. Simplified Markov Chain

And now all go well: $\tilde{F} : [0,1]^4 \rightarrow [0,1]^4$ and $\|\tilde{F}(\vec{\tau}) - \tilde{F}(\vec{\tau}')\| \leq \alpha \|\vec{\tau} - \vec{\tau}'\|$.

Finally the theorem states that there is $\vec{\tau}$ such that $\tilde{F}(\vec{\tau}) = \vec{\tau}$.

Let us show that $F(\vec{\tau}) = \vec{\tau}$. Let be i. We have $\tilde{F}(\vec{\tau})_i = \tau_i$.

Case 1: if $\tau_i > 0$, then $\tilde{F}(\vec{\tau})_i > 0$, and therefore $\tilde{F}(\vec{\tau})_i = F(\vec{\tau})_i$.

Case 2: if $\tau_i = 0$, we have $\tilde{F}(\vec{\tau})_i = 0$, and thus $F(\vec{\tau})_i \leq 0$. But because $\tau_i = 0$, we have $F(\vec{\tau})_i = b_i \geq 0$, thus $\tilde{F}(\vec{\tau})_i = F(\vec{\tau})_i = 0$.

In brief, in all cases, $F(\vec{\tau}) = \tilde{F}(\vec{\tau}) = \vec{\tau}$, and thus $f(\vec{\tau}) = \vec{b}$, and we find a solution that converges fast. To summarize we have the following Theorem:

Theorem 2. *Assume that* $\|\vec{b}\| \leq \frac{1}{m+h+1}$. *Assume that* $\alpha < 1$, *where* α *is given by* (6). *Then there is* $\vec{\tau} \in [0,1]^4$ *such that* $f(\vec{\tau}) = \vec{b}$, *and we can get a very good approximation of* $\vec{\tau}$.

IV. Conclusion

In this paper we presented a complete description on the use of the fixed point theorem method to solve the complex performance analysis problem of Wi-Fi networks. First we briefly described the underlying analytical model which is more detailed in our previous paper [10]. Then we presented step by step the long chain of equations and theorems required to solve the problem. We finally proved that for the method to converge we need the probability of error to be very big which is not the case in general. We believe that our paper constitute a basis for everyone willing to use this kind of mathematical methods to solve this problem or other similar networking problem.

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